

Dynamics (defn):— Dynamics is the branch of applied mathematics which deals with the state of motion of the bodies under the action of force.

and (ii) kinetics. Dynamics is generally divided into two parts, (i) kinematics from all considerations of force, mass or energy. Kinetics is concerned with the effect of forces on the motion of the body.

Formulae:— (i) Velocity at P is  $v = \frac{dx}{dt}$

(ii) Velocity of B relative to A = Velocity of B + reversed velocity of A.

(iii) Acceleration,  $f = \frac{d^2x}{dt^2}$

$$\text{also i.e. } f = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \quad \text{from (i)}$$

$$(iv) \text{ Also } f = \frac{dv}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx}$$

To remember the following useful notations,

$$\frac{dx}{dt} = x, \frac{du}{dt} = v, \frac{d^2x}{dt^2} = x \text{ etc.}$$

$$(v) v = u + ft, v^2 = u^2 + 2fs, s = ut + \frac{1}{2}ft^2$$

Theorem [1] (i) Define simple harmonic motion. Find its periodic time, amplitude and frequency.

(ii) Prove that the period of simple harmonic motion is independent of amplitude.

(iii) Establish the result  $T^2 \propto \text{constant}$ , where the symbols have the usual significance.

(iv) A particle moves in a straight line OA starting from rest at A and moving with an acceleration which is always directed towards O and varies as its distance from O; find the motion.

Solution: SIMPLE HARMONIC MOTION (Definition):—

A particle is said to execute simple harmonic motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the straight line and is proportional to the distance of the particle from the fixed point.

Now let us suppose that O be the fixed point in the straight line A'OA and let P be the position of the particle at any time t so that  $OP = x$ .

Let the acceleration at this distance be  $\mu x$ , where  $\mu$  is constant.

Since the acceleration  $\frac{d^2x}{dt^2}$  is in the direction of x increasing, i.e. in the direction OP, while the acceleration  $\mu x$  is towards O, i.e., in the direction PO, therefore the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu x \quad \dots \dots \quad (1)$$

$$\text{or. } v \frac{dv}{dx} = -\mu x$$

$$\text{or. } v dv = -\mu x dx$$

$$[\because \frac{dv}{dt} = f = v \frac{dx}{dt} \text{ by formulae}]$$

$$\frac{1}{2}v^2 = -\frac{\mu}{2}x^2 + \frac{C}{2}, \text{ where } \frac{C}{2} \text{ is constant}$$

$$\text{or, } v^2 = -\mu x^2 + C$$

Now let us again suppose that the particle started from rest from the point A so that  $OA = a$ , we get

$$0 = -\mu a^2 + C$$

$$\text{or, } C = \mu a^2$$

Now From ①, we get

$$v^2 = -\mu x^2 + \mu a^2$$

$$\Rightarrow v^2 = \mu(a^2 - x^2)$$

$$\Rightarrow v = \pm \sqrt{\mu(a^2 - x^2)}$$

If the particle moves from A towards O, v is negative as long as OP is positive.

$$\therefore v = -\sqrt{\mu(a^2 - x^2)}$$

$$\text{or, } \frac{dx}{dt} = -\sqrt{\mu(a^2 - x^2)}$$

$$\text{or, } \sqrt{\mu} dt = -\frac{dx}{\sqrt{a^2 - x^2}}$$

Integrating, we get

$$\sqrt{\mu} \cdot t = \cos^{-1} \frac{x}{a} + D, \text{ where } D \text{ is constant.}$$

Initially at A,  $t=0, x=a$ ,

$$\text{Hence } 0 = \cos^{-1} \frac{a}{a} + D$$

$$\text{or } 0 = \cos^{-1} 1 + D = 0 + D \therefore D = 0$$

$\therefore$  putting,  $D=0$ , we get

$$\mu t = \cos^{-1} \frac{x}{a}$$

$$\text{or, } x = a \cos(\mu t)$$

②

(3)

which gives the distance of the particle from O at any time  $t$ . If  $x=0$ , i.e. when the particle arrives at O then by ② the velocity  $= -\sqrt{\mu t}$ . The particle thus passes on the other side of O and its velocity will go on decreasing as the acceleration alters its direction. The particle comes to rest at a point A' such that  $OA = OA'$ . But the particle is being attracted towards O and reaches O with velocity zero due to which it passes O and moves towards A again stops at A. The whole motion of the particle is thus over again. Now putting  $x=0$  in ③, we get the time from A to O,

$$\text{i.e. } 0 = a \cos(\sqrt{\mu} \cdot t)$$

$$\text{or, } \sqrt{\mu} \cdot t = \frac{\pi}{2}$$

$$\text{or, } t = \frac{\pi}{2\sqrt{\mu}}$$

Therefore, the time  $T$  of one complete oscillation = the time from A to A' = 4 times from A to O and back again.

$$\therefore T = 4 \times \frac{\pi}{2\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}} \text{ or, } T^2 \mu = 4\pi^2 = \text{constant}$$

which is called the periodic time. Hence the result

Thus we have that the periodic time is independent of the amplitude  $a$ .

Frequency: - Frequency is the number of one complete oscillation in one second and hence frequency =  $\frac{1}{\text{periodic time}} = \frac{\sqrt{\mu}}{2\pi}$

$$\text{i.e. } n = \frac{\sqrt{\mu}}{2\pi} \text{ or, } \mu = 4\pi^2 n^2$$

where  $n$  denotes the frequency.

Hence theorem proved